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On the disruption of Whitney's lemma for simply connected 4-manifolds (in piecewise-linear and homotopy versions)

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Whitney's lemma [7] states that intersection points of smooth n-submanifolds of a simply connected 2n-manifold can be eliminated if the intersection number of the two submanifolds is equal to zero and $2n \geqq 6$. However, this lemma fails for $2n = 4$. This was first pointed out by Kervaire and Milnor [2] who found 2-dimensional homology classes ξ_1, ξ_2 of a simply connected 4-manifold such that (i) ξ_1 and ξ_2 are represented by smoothly embedded 2-spheres, (ii) the intersection number $\xi_1 \cdot \xi_2 = 0$ but (iii) there are no smoothly embedded disjoint 2-spheres which represent ξ_1, ξ_2 respectively. However, one can easily verify that their classes ξ_1, ξ_2 can be represented by disjoint piecewise-linearly (PL) embedded 2-spheres (with locally knotted points).

In this paper we shall give an example (Example 1) which shows that it is not always possible to represent two homology classes ξ_1, ξ_2 with $\xi_1 \cdot \xi_2 = 0$ by disjoint PL embedded 2-spheres. We shall also give an example (Example 2) in which one cannot represent a homology class ξ with $\xi \cdot [S_i] = 0$ ($\{S_i\}$ being a finite set of embedded 2-spheres) by a continuous map of a

2-sphere whose image is disjoint of these 2-spheres $\{S_i\}$.

§1. The PL case.

EXAMPLE 1. There exists a compact 1-connected 4-manifold W^4 (with boundary) which satisfis the following conditions : (i) There are two primitive homology classes ξ_1, $\xi_2 \in H(W^4 ; \mathbb{Z})$ with $\xi_1 \cdot \xi_2 = 0$, but (ii) one cannot represent ξ_1, ξ_2 by PL embedded 2-spheres with disjoint images.

We start with the following link :

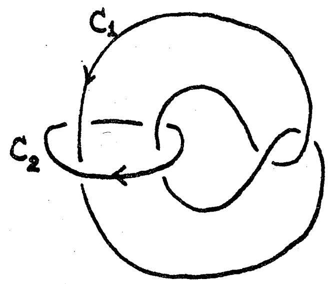

Fig. 1.

Since each of the components C_1, C_2 is a trivial knot, it has a trivial framing in S^3: $C_1 \times D^2$, $C_2 \times D^2$. Attach 2-handles h_1, h_2 to D^4 along these trivially framed circles. Then we obtain the 1-connected 4-manifold W^4 with boundary. Clearly $H_2(W ; Z) = \mathbb{Z} \oplus \mathbb{Z}$ of which each summand is generated by the respective 2-handles. Let ξ_1, ξ_2 be the two generators.

LEMMA 1. Suppose that ξ_1 (or ξ_2) is represented by a PL embedded 2-sphere Σ^2 which has a singular point (*i.e.*, a locally knotted point) of knot type k (Cf. Fox and Milnor [1]).

<u>Then $\varphi(k)=0$, where $\varphi(k)$ denotes the Robertello invariant of the knot k. (See Robertello [3].)</u>

LEMMA 2. <u>Suppose that $\xi_1+\xi_2$ is represented by a PL embedded 2-sphere Σ^2 with a singular point of knot type k. Then $\varphi(k)=1$.</u>

These lemmas will be proved later. Since the linking number of our link is equal to zero, the intersection number $\xi_1\cdot\xi_2=0$. Now we shall show that ξ_1, ξ_2 cannot be represented by disjoint PL embedded spheres. Otherwise, we would have two 2-spheres Σ_1, Σ_2 ($\subset W^4$) which represent ξ_1, ξ_2 respectively. By Lemma 1, the singularities k_1, k_2 of these 2-spheres have Robertello invariant zero. We take the connected sum of these two spheres and would obtain a PL embedded 2-sphere $\Sigma_1\#\Sigma_2$ ($\subset W^4$) which represents $\xi_1+\xi_2$ and whose singularity has Robertello invariant $\varphi(k_1)+\varphi(k_2)=0$. This contradicts Lemma 2.

<u>Proof of Lemma 1.</u> We shall prove the lemma for ξ_1. The proof for ξ_2 is the same. Suppose ξ_1 is represented by a PL embedded 2-sphere Σ^2 with a singularity k. Let D_1, D_2 be transverse disks of the attached 2-handles h_1, h_2 (i.e. cocores in the terminology of Rourke and Sanderson [4, p.74]). We may assume that Σ^2 intersects D_1, D_2 transverselly with algebraic intersection numbers 1, 0, respectively. Let U_1, U_2 be (sufficiently thin) tubular neighbourhoods of D_1, D_2 in W^4. Then $V^4=\overline{W^4-(U_1\cup U_2)}$ is PL-homeomorphic with a 4-disk, and on the boundary of V^4 we have a link $\ell=\Sigma^2\cap(\partial U_1\cup\partial U_2)$. observe

that one can obtain the link ℓ starting with the (trivial) knot C_1 (Fig.2) or with the link of Fig.3 by adding a finite number of (O,L,K)-pairs in Tristram's sense ([6], Def.3.1), where L is the knot C_1 or the link of Fig.3 and K is any component of L. (This construction of ℓ will be referred to as the explicit construction.) Thus ℓ is a proper link in the sense of Robertello [3, p.546]. ℓ is clearly related (in Robertello's sense [3, p.547]) to the singularity knot k. Since ℓ is a proper link, the Robertello invariant of a knot which is related to ℓ depends only on ℓ. Therefore, we can compute $\varphi(k)$ by any knot which is related to ℓ ([3],Th.2). However, from the explicit construction of ℓ it is easily verified that ℓ is related to a trivial knot C_1. This implies that $\varphi(k)=0$.

C_1

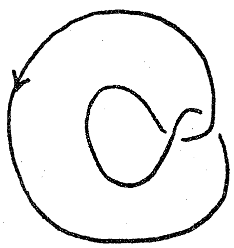

Fig.2

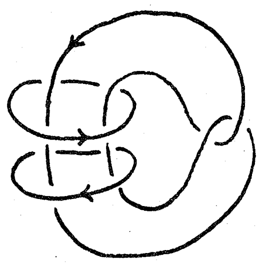

Fig.3

Q.E.D.

Proof of Lemma 2. Let Σ^2 be a PL embedded 2-sphere ($\subset W^4$) which represents $\xi_1+\xi_2$. Then Σ^2 intersects D_1, D_2 with algebraic intersection numbers 1,1.

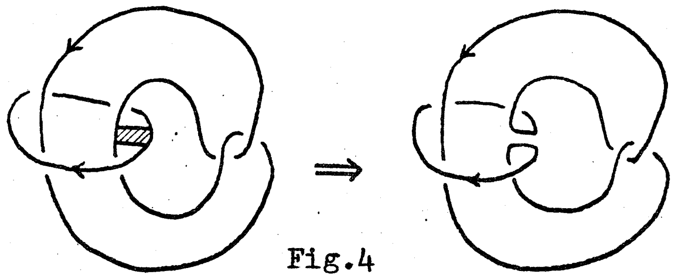

Fig.4

Thus, by the same reasoning as the previous proof, the link $\ell = \Sigma^2 \cap (\partial U_1 \cup \partial U_2)$ $(\subset \partial V)$ is proper and is related to the link of Fig.1. The link of Fig. 1 is related to a trefoil 3_1 (See Fig.4).

Since $\varphi(3_1)=1$, we know that the singularity k of Σ^2, which is also related to ℓ, has Robertello invariant 1.

Q.E.D.

PROBLEM 1. Find a closed example with the same property.

PROBLEM 2. Determine whether ξ_1, ξ_2 are represented by topologically embedded 2-spheres with disjoint images.

§2. The homotopy case.

EXAMPLE 2. There exists a closed 1-connected 4-manifold M^4 with the following properties : (i) There are smoothly embedded 16 2-spheres $S_1, \cdots, S_{16}$ with disjoint images, (ii) there is a continuous map $f: S^2 \to M^4$ of a 2-sphere to the manifold with $(f_*[S^2])\cdot[S_i^2]=0$ for $i=1, \cdots, 16$, but (iii) f cannot be homotopic to any map $g: S^2 \to M^4$

with $g(S^2)\cap(\bigcup_{i=1}^{16} S_i^2)=\phi$.

The manifold M^4 is, in fact, a Kummer manifold (Cf. Spanier[5]). Let us recall the construction. We take a 4-dimensional torus $T^4=S^1\times S^1\times S^1\times S^1$ and consider the involution σ defined by $\sigma(z_1,z_2,z_3,z_4)=(\bar{z}_1,\bar{z}_2,\bar{z}_3,\bar{z}_4)$, where we are considering $S^1=\{z\in\mathbb{C};\ |z|=1\}$. Then σ has 16 fixed points $P_1,\cdots,P_{16}$. The quotient variety T^4/σ has thus 16 singular points each of which locally looks like a cone over a 3-dimensional (real) projective space. Blow up these singularities, in other words, delete small regular neighbourhoods of the singular points and glue copies of the total space E of a 2-disk bundle over S^2 with Euler class -2. Then we obtain a closed smooth 4-manifold M^4 which contains 16 smoothly embedded 2-spheres (as exceptional curves or zero-sections of E' s) . Denote these spheres by $S_1^2,\cdots,S_{16}^2$. Note that $[S_i^2]\cdot[S_j^2]=-2\delta_{ij}$ (Kronecker's delta). It is known that the second betti number $b_2(M^4)=22$ (cf.[5]). Thus we have a non-zero homology class $\xi\in H_2(M^4;Z)$ such that $\xi\cdot[S_i^2]=0$ ($\forall i=1,\cdots 16$). Since M^4 is 1-connected ([5]), $H_2(M^4;\mathbb{Z})=\pi_2(M^4)$. Hence ξ is represented by a continuous map $f:S^2\to M^4$. Suppose $f\simeq g$ with $g(S^2)\cap(\bigcup_{i=1}^{16}S_i^2)=0$. Then, since $M^4-\bigcup_{i=1}^{16}S_i^2=T^4/\sigma$ -(the 16 points), the map g would be lifted to $\tilde{g}: S^2\to T^4$-(the 16 points). However, $\pi_2(T^4\text{-16 points})=\{0\}$. This implies that $g\simeq o$, which contradicts $\xi \neq 0$. Q.E.D.

PROBLEM 3. <u>Find a similar example with a smaller number of spheres.</u>

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